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Application of the reverberation-ray matrix to the propagation of elastic waves in a layered solid

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Abstract

This paper extends the newly developed method of reverberation-ray matrix (J. Sound Vibrat. 230(4) (2000) 743) to the propagation of elastic waves in a layered solid. The steady state waves generated by point source (axisymmetric problem) or a line source (plane strain problem) are expressed by the Sommerfield-weyl integrals of wave numbers. The waves radiated from the source are reflected or refracted at the interface of two adjacent layers, and the process of transmission and reflection is represented by a local scattering matrix; and the process of wave transmitting from one interface to the neighbouring one is represented by a local phase matrix. The local matrices of all layers are then stacked to form the global scattering matrix and global phase matrix of the layered medium separately. The product of these two matrices together with a global permutation matrix gives rise to the reverberation-ray matrix **R**, which represents the multi-reflected and transmitted steady state waves within the entire medium. The transient waves are then determined by another integration over the frequency, and the integrand of the double integral in frequency and wave number, known as the ray-integrals, contains a power series of **R**. The ray-integrals so formulated are particularly suitable for evaluating the transient waves involving a large number of generalized-rays by calculating the double integrals numerically as illustrated by the example of a laminated plate in this paper.

Keywords: Elastic waves; Reverberation-ray matrix; Layered solid

1. Introduction

The study of waves in layered media was initiated in early 1940s by Ewing et al. (1948) in connection with the project to measure and analyze the propagation of sound in shallow water of the US Atlantic Coast. Their memoir which was published in 1948 contains an article by Pekeris who models the vertically stratified sea water including the sandy bottom by a layered liquid with homogeneous layers of unequal thickness. The propagation of transient waves is then represented by a double integral of the steady state wave function, one with respect to the radial wave number κ and the other with respect to the circular

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frequency ω. The integration can be carried out in two procedures, one is called the normal mode method, the other the ray method. Their method of observations, theory and analysis were soon applied to the study of elastic waves in layered solid as summarized in the monograph by Ewing et al. (1957), and another one by Brekhovskikh (1960).

In the integrand of double integral, the steady state wave function is composed of three parts, a harmonic function in time t, a function depending on the transverse coordinate x and another one depending on the depth coordinate z. The depth function which is the solution of ordinary differential equations in variable z, consists the product sinusoidal (hyperbolic) functions with unknown coefficients, two for sound wave in each layer of liquid and four for axisymmetric wave in each layer of a solid. Various methods have been proposed, including the method of transfer matrix (Thomson, 1950; Haskell, 1953), to determine these unknown coefficients from the conditions of continuity at the interface of adjacent layers. They can all be considered as special cases of method of propagator matrix which was proposed earlier by Volterra to solve ordinary differential equations with variable coefficients (Gilbert and Backus, 1966). Once the unspecified coefficients are fixed, the double integration in κ and ω can be carried out by a variety of procedures as discussed by Kennett (1983).

In this paper, we propose an alternative to the method of propagation matrix, called the method of reverberation-ray matrix. The method originated from a study of wave propagation in a frame structure, which is made of slender structural members, connected at both ends by pins or rigid connectors (Howard and Pao, 1998; Pao et al., 1999). The method has been modified for investigating the propagation of sound waves in a layered liquid (Pao et al., 2000), and the modified one is extended to study elastic waves in a layered solid in this paper.

We discuss the formulation of the reverberation matrix in next three sections. A set of local coordinates is introduced for each layer. The wave potential in each layer is transformed into the spectral domain by applying double Fourier transforms for plane problems or Fourier–Hankel transform for axisymmetric problems (Section 2). In the spectral domain, the local scattering matrix to transfer arrival waves to departure waves at an interface or boundary is derived from the conditions of continuity of stresses and displacements at the interface (Section 3). The reverberation matrix which represents the waves reverberating in multilayered medium is then formed from the product of global scattering matrix $\mathbf{S}$, phase matrix $\mathbf{P}$ and permutation matrix $\mathbf{U}$, that is $\mathbf{R} \equiv \mathbf{SPU}$. The unknown coefficients of the depth function in the integrand are then expressed in terms of the matrix product $[\mathbf{I} - \mathbf{R}]^{-1}\mathbf{s}$ where the column matrix $\mathbf{s}$ represents the waves radiated from each and every point source in the layered medium (Section 4).

In Section 5, transient waves observed at different receivers are determined by Fourier synthesizing the steady state waves over all frequency and transverse wave numbers (inverse Fourier transform or inverse Fourier Hankle transform). The inverse of the matrix $[\mathbf{I} - \mathbf{R}]$ in the integrand of the double integral is replaced by the power series $[\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \cdots + \mathbf{R}^N + \cdots]$ through the Neumann expansion, and the original double integral which is singular at the poles of $\det[\mathbf{I} - \mathbf{R}] = 0$ is then converted into a series of double integrals, known as the ray-integrals, each being regular within the range of integration. Physically, the ray-integrals represent a group of partial waves which are transmitted from the source to a receiver along specific ray paths of multiple reflections and transmissions, the first term $\mathbf{I}s$ being the direct way from the source to receiver, the second term $\mathbf{R}s$ being the ray that is reflected or transmitted (scattered) once at the interface, the third term $\mathbf{R}^2 s$ being scattered twice, etc. The ray-integrals so derived are identical to those formulated by intuition in the method of generalized-rays (Pao and Gajewski, 1977), each integral can then be evaluated by applying Cagniard's de-Hoop method, or numerical integration.

In Section 6, we show the application of the method to the evaluation of transient waves in the laminated plate via the example of a three-layered plate. The waves are generated by a point force on the surface, and two receivers are placed at opposite sides of the plate. Since the ray-integrals do not have poles of singularity, the double integrations in frequency and wave number are accomplished by direct numerical integration with a computer code for inverse Hankle transform and another one (FFT) for inverse Fourier transform. The advantages of applying the method of reverberation matrix and the accuracy of the transient waves so calculated are discussed in the final section of conclusion.

2. Elastic waves in a layered solid

2.1. Plane strain problems

Consider a multilayered solid separated by parallel planes $z = Z^J$ $(J = 0, 1, 2, \dots, N)$, where the coordinate system (x, z), in which the wave normals of the P-wave and S-wave lie, is selected and shown in Fig. 1. The plane-strain problem and anti-plane problem can be investigated separately. All sources are assumed to locate at $(0, z)$ on an interface between two layers. If the source is situated at interior of a layer, then an additional interface passing the layer is added artificially to divide the original layer into two portions of equal material properties. We shall designate the plane interface with $I, J, K, \dots$ and layers with two capital letters. All physical quantities at the interface $z = Z^J$ will carry the superscript J; those at the layer bounded by two adjacent interfaces $z = Z^J$ and $z = Z^K$ carries two superscripts JK. Thus, the mass density, elasticity modulus in the layer are denoted by ρ^{JK}, λ^{JK} and μ^{JK}; the force applied at the interface $z = Z^J$ by vector $\mathbf{f}^J$.

In this section, however, we shall omit all superscripts for wave quantities with each layer and only discuss the plane-strain problem. Similar results will be obtained for the anti-plane problem.

For plane strain problem, the displacement field $\mathbf{u} = [u_x, 0, u_z]$ and the stress components $(\sigma_{xx}, \sigma_{xz}, \sigma_{zz})$ are determined respectively from the wave potentials $\varphi(x, z, t)$ and $\psi(x, z, t)$ by the relations

$$u_x = \frac{\partial \varphi}{\partial x} - \frac{\partial \psi}{\partial z}, \quad u_z = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi}{\partial x} \tag{1}$$

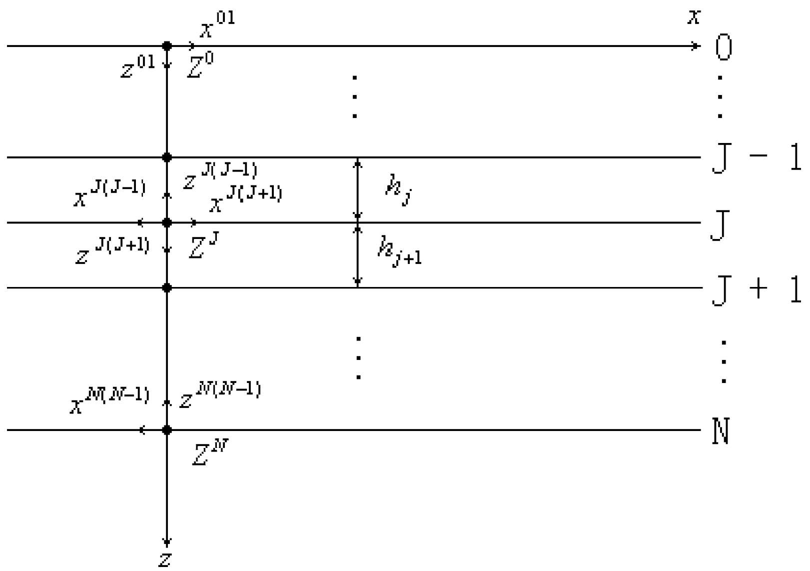

Fig. 1. Geometry and coordinates in a layered solid.

and

$$\sigma_{xx} = \lambda\nabla^2\varphi + 2\mu\left[\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\psi}{\partial x\partial z}\right],$$
$$\sigma_{xz} = \mu\left[2\frac{\partial^2\varphi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right], \tag{2}$$
$$\sigma_{zz} = \lambda\nabla^2\varphi + 2\mu\left[\frac{\partial^2\varphi}{\partial z^2} + \frac{\partial^2\psi}{\partial x\partial z}\right].$$

The wave functions $\varphi(x,z,t)$ and $\psi(x,z,t)$ satisfy the wave equations respectively

$$\nabla^2\varphi = \frac{1}{c_p^2}\frac{\partial^2\varphi}{\partial t^2}, \quad \nabla^2\psi = \frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}, \tag{3}$$

where $c_p = [(\lambda + 2\mu)/\rho]^{1/2}$ is P-wave speed, and $c_s = [\mu/\rho]^{1/2}$ is S-wave speed, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$.

The Fourier transform of the function $f(x,z,t)$ in time variable t and the inverse Fourier transform of $F(x,z,\omega)$ are given by

$$F(x,z,\omega) = \int_{-\infty}^{\infty} f(x,z,t)e^{i\omega t}\,dt, \tag{4a}$$

$$f(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x,z,\omega)e^{-i\omega t}\,d\omega. \tag{4b}$$

Similarly, the Fourier transform of $F(x,z,\omega)$ in space variable x and the inverse Fourier transform of $\hat{f}(k,z,\omega)$ are given by

$$\hat{f}(k,z,\omega) = \int_{-\infty}^{\infty} F(x,z,\omega)e^{-ikx}\,dx, \tag{5a}$$

$$F(x,z,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k,z,\omega)e^{ikx}\,dk. \tag{5b}$$

Apply inverse Fourier transform in time variable $t(\omega)$ and spatial variable $x(k)$ to obtain the solutions for the potentials φ and ψ

$$\varphi(x,z,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\varphi}(k,z,\omega)e^{i(kx-\omega t)}\,dk\,d\omega, \tag{6a}$$

$$\psi(x,z,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}(k,z,\omega)e^{i(kx-\omega t)}\,dk\,d\omega, \tag{6b}$$

where the double-transformed potentials satisfy

$$\frac{d^2\hat{\varphi}}{dz^2} + \alpha^2\hat{\varphi} = 0, \quad \alpha = (\omega^2/c_p^2 - k^2)^{1/2}, \tag{7a}$$

$$\frac{d^2\hat{\psi}}{dz^2} + \beta^2\hat{\psi} = 0, \quad \beta = (\omega^2/c_s^2 - k^2)^{1/2}. \tag{7b}$$

The solutions for the equations can be expressed respectively as

$$\hat{\varphi}(k,z,\omega) = \hat{a}_p(k,\omega)e^{-i\alpha z} + \hat{d}_p(k,\omega)e^{i\alpha z}, \tag{8a}$$

$$\hat{\psi}(k,z,\omega) = \hat{a}_s(k,\omega)e^{-i\beta z} + \hat{d}_s(k,\omega)e^{i\beta z}, \tag{8b}$$

where $\hat{a}_i$ and $\hat{d}_i$ ($i = $ p, s) are unknown coefficients. Furthermore, the twice-transformed components of the displacements and stresses are given by

$$\begin{aligned}\hat{u}_x &= ik\hat{\varphi} - \frac{d\hat{\psi}}{dz} = ik[\hat{a}_p e^{-i\alpha z} + \hat{d}_p e^{i\alpha z}] + i\beta[\hat{a}_s e^{-i\beta z} - \hat{d}_s e^{i\beta z}],\\ \hat{u}_z &= \frac{d\hat{\varphi}}{dz} + ik\hat{\psi} = i\alpha[-\hat{a}_p e^{-i\alpha z} + \hat{d}_p e^{i\alpha z}] + ik[\hat{a}_s e^{-i\beta z} + \hat{d}_s e^{i\beta z}],\end{aligned} \tag{9}$$

$$\begin{aligned}\hat{\sigma}_{xx} &= \mu\left[(k^2-\beta^2)\hat{\varphi} + 2ik\frac{d\hat{\psi}}{dz}\right] = \mu\{(k^2-\beta^2)[\hat{a}_p e^{-i\alpha z} + \hat{d}_p e^{i\alpha z}] + 2\beta k[\hat{a}_s e^{-i\beta z} - \hat{d}_s e^{i\beta z}]\},\\ \hat{\sigma}_{xz} &= \mu\left[2ik\frac{d\hat{\varphi}}{dz} + (\beta^2-k^2)\hat{\psi}\right] = \mu\{2\alpha k[\hat{a}_p e^{-i\alpha z} - \hat{d}_p e^{i\alpha z}] + (\beta^2-k^2)[\hat{a}_s e^{-i\beta z} + \hat{d}_s e^{i\beta z}]\},\\ \hat{\sigma}_{zz} &= \mu\left[(2\alpha^2-\kappa_s^2)\hat{\varphi} - 2ik\frac{d\hat{\psi}}{dz}\right] = \mu\{(2\alpha^2-\kappa_s^2)[\hat{a}_p e^{-i\alpha z} + \hat{d}_p e^{i\alpha z}] - 2\beta k[\hat{a}_s e^{-i\beta z} - \hat{d}_s e^{i\beta z}]\},\end{aligned}$$

where $\kappa_p = \omega/c_p$, $\kappa_s = \omega/c_s$.

2.2. Axisymmetric problems

For symmetric wave motion about the z-axis, the displacements are independent of the θ coordinate. Excluding the rotational motion, the displacement filed $\mathbf{u} = [u_r, 0, u_z]$ and stress components are determined from two potentials $\varphi(r,z,t)$ and $\psi(r,z,t)$ by the follows:

$$u_r = \frac{\partial\varphi}{\partial r} + \frac{\partial^2\psi}{\partial r\partial z}, \quad u_z = \frac{\partial\varphi}{\partial z} - \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}, \tag{10a}$$

$$\begin{aligned}\sigma_{rr} &= \lambda\nabla^2\varphi + 2\mu\left[\frac{\partial^2\varphi}{\partial r^2} + \frac{\partial^3\psi}{\partial r^2\partial z}\right],\\ \sigma_{rz} &= \mu\left[2\frac{\partial^2\varphi}{\partial r\partial z} + 2\frac{\partial^3\psi}{\partial r\partial z^2} - \frac{\partial}{\partial r}\nabla^2\psi\right],\\ \sigma_{zz} &= \lambda\nabla^2\varphi + 2\mu\left[\frac{\partial^2\varphi}{\partial z^2} - \frac{\partial^3\psi}{\partial z\partial r^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial z\partial r}\right],\end{aligned} \tag{10b}$$

where φ and ψ satisfy Eq. (3) with $\nabla^2 = \partial^2/\partial r^2 + r^{-1}\partial/\partial r + \partial^2/\partial z^2$.

Apply Fourier transform in time variable $t(\omega)$ and spatial variable $r(\kappa)$ to obtain the solutions for the potentials φ and ψ:

$$\varphi(r,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega\int_0^{\infty}\hat{\varphi}(\kappa,z,\omega)J_0(\kappa r)\kappa\,d\kappa, \tag{10c}$$

$$\psi(r,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\,d\omega\int_0^{\infty}\hat{\psi}(\kappa,z,\omega)J_0(\kappa r)\kappa\,d\kappa. \tag{10d}$$

The solutions for the twice-transformed potentials $\hat{\varphi}$ and $\hat{\psi}$ are similar with those in Eqs. (8a) and (8b),

$$\hat{\varphi}(\kappa,z,\omega) = \hat{a}_p(\kappa,\omega)e^{-i\alpha z} + \hat{d}_p(\kappa,\omega)e^{i\alpha z}, \tag{10e}$$

$$\hat{\psi}(\kappa,z,\omega) = \hat{a}_s(\kappa,\omega)e^{-i\beta z} + \hat{d}_s(\kappa,\omega)e^{i\beta z}. \tag{10f}$$

3. Scattering matrices for waves at interfaces

For simplification, the following discussion is limited in the case of plane strain problems. The method can be parallelly extended to the case of axisymmetric problems if the cylindrical coordinates and Hankel transform are adopted.

We restore the superscripts to all physical quantities and introduce a set of local coordinates (x^{JK}, z^{JK}) for each layer above $(K = J - 1)$ and below $(K = J + 1)$ the interface J, as shown in Fig. 1. We have

$$x^{J(J-1)} = -x^{J(J+1)} = X \quad \text{and} \quad z^{JK} = h^{JK} - z^{KJ}, \tag{11}$$

where $h^{JK} = h^{KJ}$ is the thickness of the layer JK. Within the layer JK, the two superscripts for μ, ρ and c are interchangeable. For convenience, the parameters of medium in the jth layer are also represented by letters with single subscripts, such as $\rho_j, \mu_j, h_j, \ldots$ etc.

3.1. Local scattering matrix at interface J

From Eqs. (8a) and (8b), the potentials in the two adjacent layers at interface J are expressed respectively by

$$\begin{aligned}
\hat{\varphi}^{J(J-1)}(k, z^{J(J-1)}, \omega) &= \hat{a}_{\mathrm{p}}^{J(J-1)} \mathrm{e}^{-\mathrm{i}\alpha_j z^{J(J-1)}} + \hat{d}_{\mathrm{p}}^{J(J-1)} \mathrm{e}^{\mathrm{i}\alpha_j z^{J(J-1)}}, \\
\hat{\psi}^{J(J-1)}(k, z^{J(J-1)}, \omega) &= \hat{a}_{\mathrm{s}}^{J(J-1)} \mathrm{e}^{-\mathrm{i}\beta_j z^{J(J-1)}} + \hat{d}_{\mathrm{s}}^{J(J-1)} \mathrm{e}^{\mathrm{i}\beta_j z^{J(J-1)}}, \\
\hat{\varphi}^{J(J+1)}(k, z^{J(J+1)}, \omega) &= \hat{a}_{\mathrm{p}}^{J(J+1)} \mathrm{e}^{-\mathrm{i}\alpha_{j+1} z^{J(J+1)}} + \hat{d}_{\mathrm{p}}^{J(J+1)} \mathrm{e}^{\mathrm{i}\alpha_{j+1} z^{J(J-1)}}, \\
\hat{\psi}^{J(J+1)}(k, z^{J(J+1)}, \omega) &= \hat{a}_{\mathrm{s}}^{J(J+1)} \mathrm{e}^{-\mathrm{i}\beta_{j+1} z^{J(J+1)}} + \hat{d}_{\mathrm{s}}^{J(J+1)} \mathrm{e}^{\mathrm{i}\beta_{j+1} z^{J(J+1)}}.
\end{aligned} \tag{12}$$

Associated with the time factor $\mathrm{e}^{-\mathrm{i}\omega t}$ in Eqs. (4a) and (4b), the term with unknown amplitude $\hat{d}_{\mathrm{p}}^{JK}$ represents a P-wave departing from the interface J and travelling in the positive direction of z^{JK}; and that with $\hat{a}_{\mathrm{p}}^{JK}$ represents a P-wave arriving at the interface J and traveling in the negative direction of z^{JK}. Similarly, $\hat{d}_{\mathrm{s}}^{JK}$ a SV-wave departing from the interface J and $\hat{a}_{\mathrm{s}}^{JK}$ a SV-wave arriving at the interface J. $\hat{a}_{\mathrm{p}}^{JK}$, $\hat{d}_{\mathrm{p}}^{JK}$, $\hat{a}_{\mathrm{s}}^{JK}$ and $\hat{d}_{\mathrm{s}}^{JK}$ are unknown functions of κ and ω, which shall be determined by boundary conditions and the continuity conditions at the interface.

If a point force or a vertical line force with time function $f(t)$ is placed at $x = 0$, $z = z^J$, the origin of two local coordinates, the source function may be represented by $-\delta(x - 0)\delta(z - z^J)f(t)$. The displacements (u_x, u_z) and the shear stress σ_{xz} should be continuous at the interface, but the normal stress σ_{zz} will jump across the interface. We can obtain four continuity conditions at the interface

$$\begin{aligned}
&\hat{u}_x^{J(J+1)}(k, 0, \omega) + \hat{u}_x^{J(J-1)}(k, 0, \omega) = 0, \\
&\hat{u}_z^{J(J+1)}(k, 0, \omega) + \hat{u}_z^{J(J-1)}(k, 0, \omega) = 0, \\
&\hat{\sigma}_{xz}^{J(J+1)}(k, 0, \omega) - \hat{\sigma}_{xz}^{J(J-1)}(k, 0, \omega) = 0, \\
&\hat{\sigma}_{zz}^{J(J+1)}(k, 0, \omega) - \hat{\sigma}_{zz}^{J(J-1)}(k, 0, \omega) = -F(\omega), \quad J = 1, 2, \ldots, N - 1.
\end{aligned} \tag{13}$$

Substituting Eqs. (9) and (12) into the previous equations, we obtain a set of equations for the unknown coefficients $\hat{a}_l^{J(J-1)}$, $\hat{a}_l^{J(J+1)}$, $\hat{d}_l^{J(J-1)}$ and $\hat{d}_l^{J(J+1)}$ $(l = \mathrm{p}, \mathrm{s})$, which is expressed in matrix form as follows:

$$\mathbf{A}^J \hat{\mathbf{a}}^J + \mathbf{D}^J \hat{\mathbf{d}}^J = \hat{\mathbf{g}}^J(k, \omega), \tag{14}$$

where $\hat{\mathbf{a}}^J$ and $\hat{\mathbf{d}}^J$ are unknowns vectors, $\mathbf{A}^J$ and $\mathbf{D}^J$ are 4×4 matrices, and $\hat{\mathbf{g}}^J$ is an external force vector, i.e.

$$\hat{\mathbf{a}}^J = (\hat{a}_{\mathrm{p}}^{J(J-1)}, \hat{a}_{\mathrm{s}}^{J(J-1)}, \hat{a}_{\mathrm{p}}^{J(J+1)}, \hat{a}_{\mathrm{s}}^{J(J+1)})^{\mathrm{T}},$$

$$\hat{\mathbf{d}}^J = (\hat{d}_{\mathrm{p}}^{J(J-1)}, \hat{d}_{\mathrm{s}}^{J(J-1)}, \hat{d}_{\mathrm{p}}^{J(J+1)}, \hat{d}_{\mathrm{s}}^{J(J+1)})^{\mathrm{T}},$$

$$\mathbf{A}^J = \begin{bmatrix} -k & -\beta_j & k & \beta_{j+1} \\ \alpha_j & -k & -\alpha_{j+1} & k \\ \mu_j(\beta_j^2 - k^2) & -2\mu_j k\beta_j & -\mu_{j+1}(\beta_{j+1}^2 - k^2) & 2\mu_{j+1}k\beta_{j+1} \\ -2\mu_j k\alpha_j & -\mu_j(\beta_j^2 - k^2) & 2\mu_{j+1}k\alpha_{j+1} & \mu_{j+1}(\beta_{j+1}^2 - k^2) \end{bmatrix},$$

$$\mathbf{D}^J = \begin{bmatrix} -k & -\beta_j & k & \beta_{j+1} \\ -\alpha_j & -k & \alpha_{j+1} & k \\ \mu_j(\beta_j^2 - k^2) & 2\mu_j k\beta_j & -\mu_{j+1}(\beta_{j+1}^2 - k^2) & -2\mu_{j+1}k\beta_{j+1} \\ 2\mu_j k\alpha_j & -\mu_j(\beta_j^2 - k^2) & -2\mu_{j+1}k\alpha_{j+1} & \mu_{j+1}(\beta_{j+1}^2 - k^2) \end{bmatrix},$$

$$\hat{\mathbf{g}}^J(k,\omega) = (0, 0, 0, -F(\omega))^{\mathrm{T}}.$$

Obviously, both components of the force vector vanish when no source is at the interface. Solving $\hat{\mathbf{d}}^J$ in terms of unknowns vector $\hat{\mathbf{a}}^J$ and a given source vector $\hat{\mathbf{g}}^J$, we find

$$\hat{\mathbf{d}}^J = \mathbf{S}^J \hat{\mathbf{a}}^J + \hat{\mathbf{s}}^J(k,\omega), \quad J = 1, 2, \ldots, N-1, \tag{15}$$

where

$$\mathbf{S}^J = -(\mathbf{D}^J)^{-1}\mathbf{A}^J = \begin{bmatrix} R_{jj}^{\mathrm{pp}} & R_{jj}^{\mathrm{sp}} & T_{(j+1)j}^{\mathrm{pp}} & T_{(j+1)j}^{\mathrm{sp}} \\ R_{jj}^{\mathrm{ps}} & R_{jj}^{\mathrm{ss}} & T_{(j+1)j}^{\mathrm{ps}} & T_{(j+1)j}^{\mathrm{ss}} \\ T_{j(j+1)}^{\mathrm{pp}} & T_{j(j+1)}^{\mathrm{sp}} & R_{(j+1)(j+1)}^{\mathrm{pp}} & R_{(j+1)(j+1)}^{\mathrm{pp}} \\ T_{j(j+1)}^{\mathrm{ps}} & T_{j(j+1)}^{\mathrm{ss}} & R_{(j+1)(j+1)}^{\mathrm{pp}} & R_{(j+1)(j+1)}^{\mathrm{ss}} \end{bmatrix} \tag{16}$$

and

$$\hat{\mathbf{s}}^J(k,\omega) = (\mathbf{D}^J)^{-1}\hat{\mathbf{g}}^J(k,\omega). \tag{17}$$

The matrix $\mathbf{S}^J$ is called the scattering matrix at Jth interface, the element of which relates an incident wave (arrival) to the transmitted or reflected wave (departure) in Jth interface. The $\hat{\mathbf{s}}^J$ is called the source wave vector, which represents the waves emitted by the source at the interface. The scattering of waves at the interface is shown in Fig. 2. The reflection and transmission coefficients in the matrix $\mathbf{S}^J$ are given by

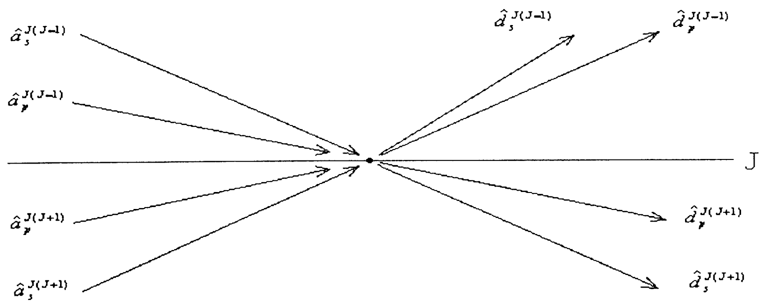

Fig. 2. Scattering of waves at the interface J.

$$
\begin{aligned}
R^{\mathrm{pp}} &= [(l_1 - l_3)(m_2 + m_4) - (l_2 + l_4)(m_1 - m_3)]/\Delta, \\
R^{\mathrm{ps}} &= 2\varepsilon(m_4 l_2 - l_4 m_2)/\Delta, \\
T^{\mathrm{pp}} &= 2(m_2 + m_4)(k l_1 - \beta_1 l_2)/k\Delta, \\
T^{\mathrm{ps}} &= 2\varepsilon(l_2 + l_4)(\alpha_1 m_3 - k m_4)/k\Delta, \\
R^{\mathrm{sp}} &= 2\varepsilon(m_3 l_1 - l_3 m_1)/\Delta, \\
R^{\mathrm{ss}} &= [(l_1 + l_3)(m_4 - m_2) - (l_4 - l_2)(m_1 + m_3)]/\Delta, \\
T^{\mathrm{sp}} &= 2\varepsilon(m_3 + m_1)(\beta_1 l_2 - k l_1)]/k\Delta, \\
T^{\mathrm{ss}} &= 2(l_1 + l_3)(\alpha_1 m_3 - k m_4)]/k\Delta,
\end{aligned}
$$

where

$$\Delta = (l_1 + l_3)(m_2 + m_4) + (l_2 + l_4)(m_1 + m_3).$$

The elements $l_1, l_2, \ldots, m_4$ may be expressed in terms of shear modulus μ_j and wave numbers k, α_i, β_i as defined in Eqs. (7a) and (7b) where the subscript $i(=1,2)$ is appended to indicate the source (1) and the adjacent (2) layer:

$$
\begin{aligned}
l_1 &= \bar{\mu}(k^2 - \beta_2^2) - 2k^2, \\
l_2 &= k/\beta_1[\bar{\mu}(k^2 - \beta_2^2) - (k^2 - \beta_1^2)], \\
l_3 &= \alpha_2/\alpha_1[(k^2 - \beta_1^2) - 2\bar{\mu}k^2], \\
l_4 &= 2k\alpha_2(1 - \bar{\mu}), \\
m_1 &= 2k\beta_2(1 - \bar{\mu}), \\
m_2 &= \beta_2/\beta_1[k^2 - \beta_1^2 - 2\bar{\mu}k^2], \\
m_3 &= k/\alpha_1[\bar{\mu}(k^2 - \beta_2^2) - (k^2 - \beta_1^2)], \\
m_4 &= \bar{\mu}(k^2 - \beta_2^2) - 2k^2,
\end{aligned}
$$

where $\bar{\mu} = \mu_2/\mu_1$. The factor ε is $+1$ or -1 according to the downgoing or upgoing incident wave.

When the media of layer J and layer $(J+1)$ are same, nonzero elements of the matrix $\mathbf{S}^J$ are only $T^{\mathrm{pp}} = T^{\mathrm{ss}} = 1$.

If the layered solid is bounded at the top, $z = Z^0 = 0$, and at the bottom, $z = Z^N$, both the wave vectors $\hat{a}$ and $\hat{d}$ degenerate into a vector with two elements. The scattering matrix $\mathbf{S}^0$ and $\mathbf{S}^N$ also degenerate into a 4×4 matrices. Particularly, in the case that the top is a free surface and the bottom is a rigid plane, they can be derived by taking appropriate limit values of density and shear wave speed for one adjacent layer, zero density and zero speed for upper layer of the free surface, and infinite density and infinite wave speed for lower layer of the rigid plane. The limit values are

$$
\begin{aligned}
\hat{\mathbf{d}}^0 &= \mathbf{S}^0\hat{\mathbf{a}}^0 + \hat{\mathbf{s}}^0(k,\omega), \\
\hat{\mathbf{d}}^N &= \mathbf{S}^N\hat{\mathbf{a}}^N + \hat{\mathbf{s}}^N(k,\omega).
\end{aligned} \tag{18}
$$

The transmission coefficients are zero and the reflection coefficients at a free surface are

$$
\begin{aligned}
R^{\mathrm{pp}} &= [4k^2\alpha\beta - (k^2 - \beta^2)^2]/\Delta, \quad R^{\mathrm{ss}} = R^{\mathrm{pp}}, \\
R^{\mathrm{ps}} &= \varepsilon 4k\alpha(k^2 - \beta^2)/\Delta, \quad R^{\mathrm{sp}} = 4\varepsilon k\beta(k^2 - \beta^2)/\Delta,
\end{aligned}
$$

where $\Delta = 4k^2\alpha\beta + \left(k^2 - \beta^2\right)^2$.

While the transmission coefficients are zero and the reflection coefficients at a fixed surface are

$$R^{\mathrm{pp}} = (\alpha\beta - k^2)/\Delta, \quad R^{\mathrm{ss}} = R^{\mathrm{pp}},$$
$$R^{\mathrm{ps}} = 2k\alpha/\Delta, \quad R^{\mathrm{sp}} = -2k\beta/\Delta,$$

where $\Delta = \alpha\beta + k^2$.

If a line of vertical forces acts at the free surface, the continuity condition will be replaced by the boundary condition, $\sigma_{zz} = -\delta(x)f(t)$ and $\sigma_{xz} = 0$ at $z = 0$. If there is no point force or line of vertical force at the rigid surface, the source wave vector $\hat{\mathbf{s}}^N$ vanishes.

If the bottom plane $z = Z^{N-1}$ is bounded by a semi-infinite space, we let the plane $z = Z^N$ recede to infinity and the thickness h_n approach infinite. From the radiation condition, the wave numbers in the semi-infinite space becomes imaginary, $a_n = \sqrt{k^2 - \omega^2/(c_{\mathrm{p}})_n^2}$ and $\beta_n = \sqrt{k^2 - \omega^2/(c_{\mathrm{s}})_n^2}$, and the wave amplitude $\hat{d}_{\mathrm{p}}^{(N-1)N}$, $\hat{d}_{\mathrm{s}}^{(N-1)N}$, $\hat{a}_{\mathrm{p}}^{N(N-1)}$ and $\hat{a}_{\mathrm{s}}^{N(N-1)}$ must vanish. The elements of the 4×4 scattering matrix $\mathbf{S}^{N-1}$ should be modified accordingly.

3.2. Global scattering matrix for the layered solid

Combining $4N$ equations in Eqs. (17) and (18), we can construct a system of equations for the entire layered solid in the following form:

$$\begin{pmatrix} \hat{\mathbf{d}}^0 \\ \hat{\mathbf{d}}^1 \\ \hat{\mathbf{d}}^2 \\ \vdots \\ \hat{\mathbf{d}}^{N-1} \\ \hat{\mathbf{d}}^N \end{pmatrix} = \begin{bmatrix} \mathbf{S}^0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{S}^1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathbf{S}^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{S}^{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{S}^N \end{bmatrix} \begin{pmatrix} \hat{\mathbf{a}}^0 \\ \hat{\mathbf{a}}^1 \\ \hat{\mathbf{a}}^2 \\ \vdots \\ \hat{\mathbf{a}}^{N-1} \\ \hat{\mathbf{a}}^N \end{pmatrix} + \begin{pmatrix} \hat{\mathbf{s}}^0 \\ \hat{\mathbf{s}}^1 \\ \hat{\mathbf{s}}^2 \\ \vdots \\ \hat{\mathbf{s}}^{N-1} \\ \hat{\mathbf{s}}^N \end{pmatrix}. \tag{19}$$

In compact notation, the previous equation is written as

$$\hat{\mathbf{d}} = \mathbf{S}\hat{\mathbf{a}} + \hat{\mathbf{s}}(k, \omega). \tag{20}$$

The vector with $4N$ elements, $\hat{\mathbf{d}}$, which is named the global departing wave vector, represents completely waves departing from all interfaces downward and upward, and the vector $\hat{\mathbf{a}}$, which is named the global arriving wave vector, represents completely waves arriving at all interfaces upward and downward. The square matrix $\mathbf{S}$ that is a block-diagonal matrix of dimension $4N$ is called the global scattering matrix. The vector $\hat{\mathbf{s}}$ with $4N$ elements, which is called the global source vector, represents waves emitted from sources located at $x = 0$, $z = Z^J$ $(J = 0, 1, 2, \ldots, N)$.

Notice that two local coordinates $(x^{J(J-1)}, z^{J(J-1)})$ and $(x^{J(J+1)}, z^{J(J+1)})$ are used to analyze waves arriving and departing from the same interface, the amplitude coefficients are treated separately from the phase functions. In this section, the elements of $\mathbf{S}^J$ represent the reflection or transmission coefficients for waves incident at the interface J are the same as those calculated from a single coordinate. Since both vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ are unknown quantities, we need an additional equation related $\hat{\mathbf{d}}$ to $\hat{\mathbf{a}}$.

4. Reverberation matrix

4.1. Phase matrix

The additional equation is supplemented by first noting that a wave departing from one side of the layer becomes the wave arriving at another side of the same layer. The amplitudes for the waves at both sides, however, differ by a phase shift factor as follows:

$$\begin{aligned} &\hat{a}_{\mathrm{p}}^{J(J-1)} = \mathrm{e}^{\mathrm{i}\alpha_j h_j}\hat{d}_{\mathrm{p}}^{(J-1)J}, \quad \hat{a}_{\mathrm{s}}^{J(J-1)} = \mathrm{e}^{\mathrm{i}\beta_j h_j}\hat{d}_{\mathrm{s}}^{(J-1)J}, \\ &\hat{d}_{\mathrm{p}}^{J(J-1)} = \mathrm{e}^{-\mathrm{i}\alpha_j h_j}\hat{a}_{\mathrm{p}}^{(J-1)J}, \ \ \hat{d}_{\mathrm{s}}^{J(J-1)} = \mathrm{e}^{-\mathrm{i}\beta_j h_j}\hat{a}_{\mathrm{s}}^{(J-1)J}, \quad J = 1,2,\ldots,N-1. \end{aligned} \tag{21}$$

Eq. (21) can be expressed by $\hat{\mathbf{a}}^j = \mathbf{P}(h_j)\hat{\mathbf{d}}^j$, where the local phase matrix is

$$\mathbf{P}(h_j) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\alpha_j h_j} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\beta_j h_j} \end{pmatrix}.$$

We introduce a new local vector at Jth interface, $\hat{\mathbf{d}}^{*j}$, and a new global vector $\hat{\mathbf{d}}^{*}$ for the departing waves as

$$\begin{aligned} &\hat{\mathbf{d}}^{*j} = (\hat{d}_{\mathrm{p}}^{(J-1)J}, \hat{d}_{\mathrm{s}}^{(J-1)J}, \hat{d}_{\mathrm{p}}^{(J+1)J}, \hat{d}_{\mathrm{s}}^{(J+1)J})^{\mathrm{T}}, \\ &\hat{\mathbf{d}}^{*} = (\hat{d}^{*0}, \hat{d}^{*1}, \hat{d}^{*2}, \ldots, \hat{d}^{*(N-1)}, \hat{d}^{*N})^{\mathrm{T}}, \end{aligned} \tag{22}$$

here $\hat{\mathbf{d}}^{*0} = \{\hat{d}_{\mathrm{p}}^{10}, \hat{d}_{\mathrm{s}}^{10}\}^{\mathrm{T}}$, $\hat{\mathbf{d}}^{*N} = \{\hat{d}_{\mathrm{p}}^{(N-1)N}, \hat{d}_{\mathrm{s}}^{(N-1)N}\}^{\mathrm{T}}$. Furthermore, all elements of wave vector $\hat{\mathbf{a}}$ are related to those of the vector $\hat{\mathbf{d}}^{*}$ as

$$\hat{\mathbf{a}} = \mathbf{P}(\kappa, h, \omega)\hat{\mathbf{d}}^{*}, \tag{23}$$

where total phase shift matrix $\mathbf{P}(k,h,\omega)$ or $\mathbf{P}(h)$ $(4N \times 4N)$ is a block-diagonal matrix, which is given by

$$\mathbf{P}(h) = \begin{bmatrix} \mathbf{P}(h_1) & 0 & & & \\ & & & 0 & \\ 0 & \mathbf{P}(h_1) & & & \\ & & \ddots & & \\ & 0 & \mathbf{P}(h_n) & 0 & \\ & & 0 & \mathbf{P}(h_n) \end{bmatrix}. \tag{24}$$

4.2. Permutation matrix

The global vectors $\hat{\mathbf{d}}^{*}$ and $\hat{\mathbf{d}}$ contain the same elements but sequenced being different order. We may express this equivalence through a permutation matrix $\mathbf{U}$,

$$\hat{\mathbf{d}}^{*} = \mathbf{U}\hat{\mathbf{d}}, \tag{25}$$

where $\mathbf{U}$ is a $4N \times 4N$ block-diagonal matrix composed of N same 4×4 sub-matrix $\mathbf{u}$ and other vanishing elements as

$$\mathbf{U} = \begin{bmatrix} \mathbf{u} & 0 & \cdots & 0 \\ 0 & \mathbf{u} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{u} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

4.3. Reverberation matrix

Substituting Eq. (25) into Eq. (23), we find the second equation that relates the vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ as

$$\hat{\mathbf{a}} = \mathbf{P}(h)\mathbf{U}\hat{\mathbf{d}}. \tag{26}$$

Solving Eqs. (20) and (26) simultaneously, we finally obtain

$$\hat{\mathbf{d}} = [\mathbf{I} - \mathbf{R}]^{-1}\hat{\mathbf{s}}(k, \omega), \tag{27}$$

$$\hat{\mathbf{a}} = \mathbf{P}(h)\mathbf{U}[\mathbf{I} - \mathbf{R}]^{-1}\hat{\mathbf{s}}(k, \omega), \tag{28}$$

where we have introduced the reverberation matrix $\mathbf{R}$ defined by

$$\mathbf{R}(k, \omega) = \mathbf{SP}(h)\mathbf{U}. \tag{29}$$

The matrix $[\mathbf{I} - \mathbf{R}(k, \omega)]^{-1}$ relates the response of the layered medium to the excitation $\hat{\mathbf{s}}(k, \omega)$ in the frequency–wave number domain. The dispersion relation for the resonant waves in the layered medium is given by

$$\det[\mathbf{I} - \mathbf{R}(k, \omega)] = 0. \tag{30}$$

The determinant in Eq. (30) which is based on evaluation of scattering waves at each interface is different from that derived by Thomson (1950) and Haskell (1953) method which is based on evaluation of the wave propagating from one interface to another. They should, however, yield the same numerical results for the dispersion relations in ω–k plane.

The frequency response for monochromatic waves in the layered medium is determined by completing inverse Fourier transform in Eqs. (6a) and (6b), after substituting of $\hat{\mathbf{a}}$ and $\hat{\mathbf{d}}$ into Eq. (9). The transient response of the same medium is determined by completing inverse Fourier transform.

5. Transient waves in the layered medium

Once the coefficient vectors $\hat{\mathbf{d}}$ and $\hat{\mathbf{a}}$ are known from Eqs. (27) and (28), the complete list of displacements and stresses in frequency domain will be expressed as

$$\widehat{\mathbf{W}}(k, \mathbf{z}, \omega) = \mathbf{B}(k, \mathbf{z}, \omega)][\mathbf{I} - \mathbf{R}(k, \omega)]^{-1}\hat{\mathbf{s}}(k, \omega), \tag{31}$$

where

$$\widehat{\mathbf{W}}(k, \mathbf{z}, \omega) = \{\hat{\mathbf{w}}(k, z^{01}, \omega), \hat{\mathbf{w}}(k, z^{10}, \omega), \dots, \hat{\mathbf{w}}(k, z^{(N-1)N}, \omega), \hat{\mathbf{w}}(k, z^{N(N-1)}, \omega)\}^{\mathrm{T}},$$

$$\hat{\mathbf{w}} = (\hat{u}_x, \hat{u}_z, \hat{\sigma}_{xz}, \hat{\sigma}_{zz})^{\mathrm{T}}, \quad z^{J(J-1)} = h_j - z^{(J-1)J},$$

and vertical coordinate vector of receivers

$$\mathbf{z} = \{z^{01}, z^{10}, z^{12}, z^{21}, \dots, z^{(N-1)N}, z^{N(N-1)}\}^{\mathrm{T}}. \tag{32}$$

The matrix $\mathbf{B}$, which is called the receiver matrix, is combined by the coefficients of the wave vectors in Eq. (9).

Applying twice inverse Fourier transforms for variables x and t, we can thus obtain the transient responses at N receivers to the sources, i.e.

$$\mathbf{W}(x, \mathbf{z}, t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \int_{-\infty}^{\infty} \widehat{\mathbf{W}}(k, \mathbf{z}, \omega)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \tag{33}$$

It is assumed that no two receivers are located in same layer. We rewrite Eq. (33) in detail as

$$\mathbf{W}(x,\mathbf{z},t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\int_{-\infty}^{\infty}\mathbf{B}(k,\mathbf{z},\omega)[\mathbf{I}-\mathbf{R}(k,\omega)]^{-1}\hat{\mathbf{s}}(k,\omega)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \tag{34}$$

For each element of wave vectors in Eq. (34), the double integral representation of the wave displacements and stresses can be calculated by either the traditional spectra method or the ray method, both first were proposed by Pekeris (1955).

5.1. Generalized ray-integrals

The method of reverberation-ray matrix is particularly suitable for ray method, if we expand the transfer function in a Neumann series

$$[\mathbf{I}-\mathbf{R}]^{-1}=\mathbf{I}+\mathbf{R}+\mathbf{R}^2+\cdots+\mathbf{R}^M+[\mathbf{I}-\mathbf{R}]^{-1}\mathbf{R}^{M+1}. \tag{35}$$

Substituting Eq. (35) into Eq. (34),

$$\mathbf{W}(x,\mathbf{z},t)=\sum_{m=0}^{M}\mathbf{W}^{(m)}(x,\mathbf{z},t)+\mathbf{W}_R^{(M+1)}(x,\mathbf{z},t), \tag{36}$$

where

$$\mathbf{W}^{(m)}(x,\mathbf{z},t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega\int_{-\infty}^{\infty}\mathbf{B}(k,\mathbf{z},\omega)\mathbf{R}^m\hat{\mathbf{s}}(k,\omega)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \tag{37}$$

and

$$\mathbf{W}^{(M+1)}(x,\mathbf{z},t)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t}\int_{-\infty}^{\infty}\mathbf{B}(k,\mathbf{z},\omega)[\mathbf{I}-\mathbf{R}]^{-1}\mathbf{R}^{M+1}\hat{\mathbf{s}}(k,\omega)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k. \tag{38}$$

The integrals in the summation can thus be evaluated term by term to obtain the "generalized-ray solution". $\mathbf{W}^{(0)}(x,\mathbf{z},t)$, the integrand of which have factor $\mathbf{R}^0$, contains the waves originally generated by the applied forces, which propagate away from the sources to the receivers at (x,z). $\mathbf{W}^{(1)}(x,\mathbf{z},t)$, the integrand of which has factor $\mathbf{R}$, contains the first set of reflections and transmissions of the source waves in the layered solid. In general, $\mathbf{W}^{(m)}(x,\mathbf{z},t)$, the integrand of which have factor $\mathbf{R}^m$, contains the set of m times reflections and transmissions of the sources waves in the layered solid. The double ray integrals can be calculated by applying the Cagniard method or by numerical integration (Cagniard, 1939; Müller, 1968, 1969; Pao and Gajewski, 1977).

To illustrate the ray paths for waves propagating from the source to a receiver, we show in Fig. 3 the $\mathbf{R}^0,\mathbf{R}^1,\mathbf{R}^2,\mathbf{R}^3,\mathbf{R}^4,\mathbf{R}^5$ groups in the top three layers of layered medium. The source which generated both P-waves and S-waves is applied at the origin and a receiver is situated somewhere in the middle of first layer. The Fig. 4a shows the first group $\mathbf{R}^0$ and second group $\mathbf{R}^1$, the latter being reflected once at the free surface. Fig. 4b, c and d show the third group $\mathbf{R}^2$, the fourth group $\mathbf{R}^3$ and the fifth group $\mathbf{R}^4$ respectively. Fig. 4e shows the sixth group $\mathbf{R}^5$ that is composed of four subgroups, each ray in the group being reflected or diffracted (transmitted) five times. Since there is a mode conversion, P to S or S to P, at each reflection or diffraction, there are $4\times 2^6=256$ rays in this group. The partial wave along each ray path in a group is represented by one ray integral in Eq. (33), which can be evaluated numerically.

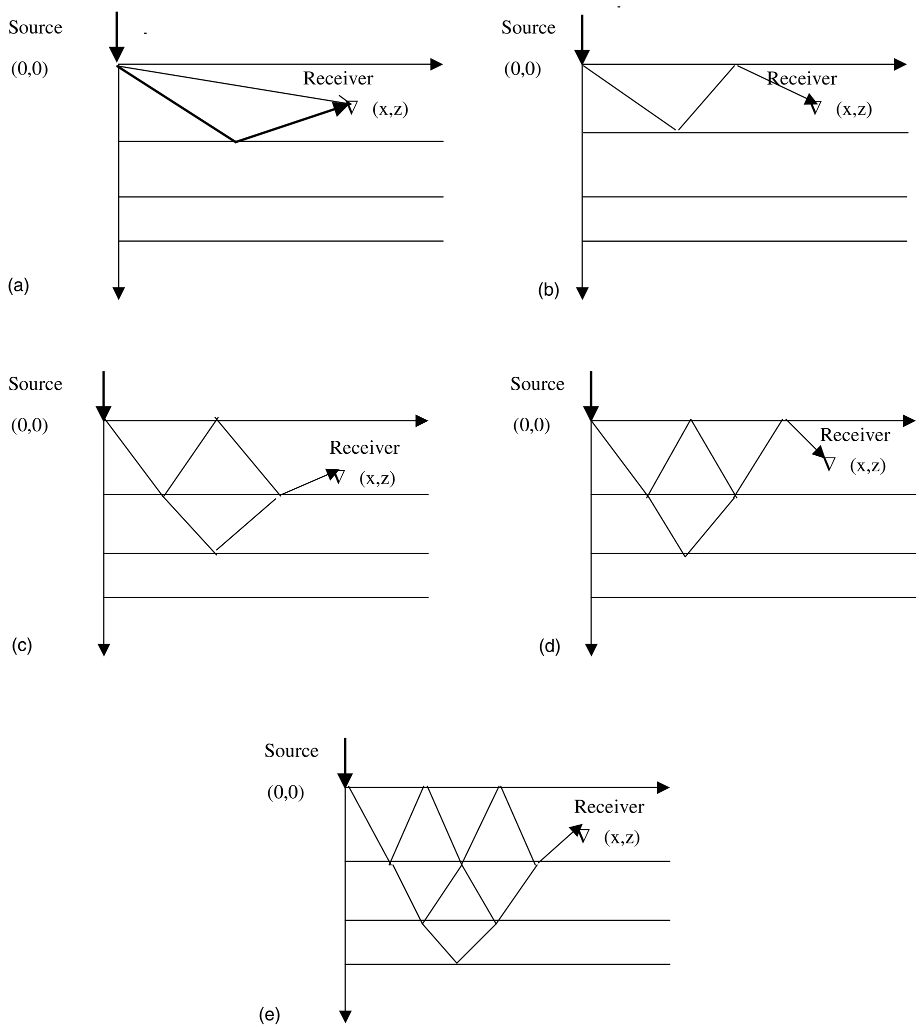

Fig. 3. Rays with reverberations: (a) first ray group for $\mathbf{R}^0$ and second ray group for $\mathbf{R}^1$, (b) third ray group for $\mathbf{R}^2$, (c) fourth ray group for $\mathbf{R}^3$, (d) fifth ray group for $\mathbf{R}^4$, (e) sixth ray group for $\mathbf{R}^5$.

5.2. Fast Hankel transform and fast Fourier transform

The double-integrals for the generalized rays are computed by applying the fast Hankel transform and the fast Fourier transform for axisymmetric problems. The fast Fourier transform is known. Here we introduce the fast Hankel transform only.

The Hankel transform is defined as

$$f(r) = \int_0^\infty \hat{f}(k) J_0(kr) k \, \mathrm{d}k. \tag{39}$$

Let $r = r_0 \mathrm{e}^x$, $k = k_0 \mathrm{e}^{-y}$, then substitute them into the formula (39), we have

$$f(r_0 \mathrm{e}^x) = k_0^2 \int_{-\infty}^\infty \mathrm{e}^{-2y} \hat{f}(r_0 \mathrm{e}^{-y}) J_0(r_0 k_0 \mathrm{e}^{x-y}) \, \mathrm{d}y. \tag{40}$$

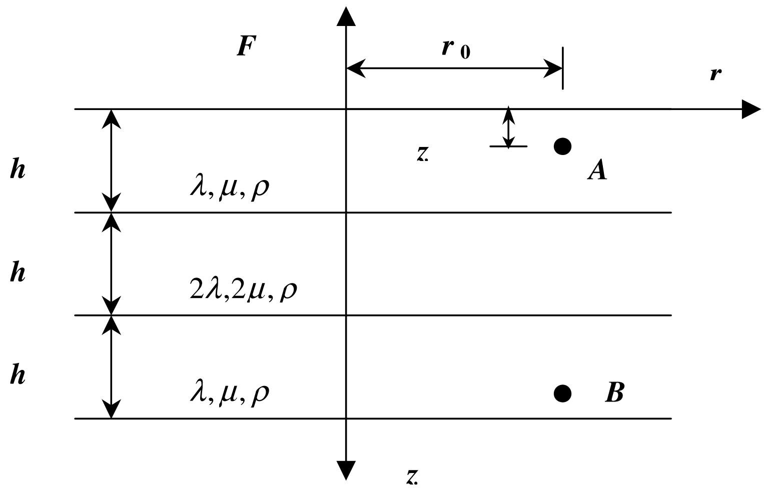

Fig. 4. A laminate with a source at (0, 0) and receivers at A (2.39 h, 0.05 h) and B (2.39 h, 2.95 h).

The right side of the above formula is considered as a convolution of two functions $e^{-2y}\hat{f}(r_0 e^{-y})$ and $k_0^2 J_0(r_0 k_0 e^x)$, which can be calculated approximately by the numerical algorithm based on FFT. Thus we can obtain N discrete samples approximately as

$$f(r_0 e^{j\delta}) = \sum_{m=0}^{N} b_m c_{j-m}, \quad j = 0, \ldots, N-1, \tag{41}$$

where N is the integer power of 2, $x, y = j\delta$, $j = 0, \ldots, N-1$.

$$b_m = \begin{cases} 0, & m = -N, \ldots, -1, \\ e^{-2m\delta}\hat{f}(r_0 e^{-m\delta}), & m = 0, \ldots, N-1, \end{cases} \quad \text{and} \quad c_{m-j} = k_0^2 \delta J_0(r_0 k_0 e^{(m-j)\delta}).$$

According to Sample theorem and accuracy request, parameters r_0, k_0, N, δ will be chosen. In general, we can choose $r_0 = r_{\max}$, $r_{\min} = e^{-(N-1)\delta}$, $k_0 = k_{\max}$, $k_{\min} = k_0 e^{(N-1)\delta}$, where $r_{\min}$ and $r_{\max}$ are minimum and maximum of r, $k_{\min}$ and $k_{\max}$ are minimum and maximum of k. They must satisfy the constrain conditions: $r_{\max} k_{\min} \delta < \pi$ and $r_{\min} k_{\max} \delta < \pi$. For the inverse Hankel transform of the integral Eqs. (10c) and (10d) can be computed by using FHT. Based on the above algorithm, a program is made to compute transient response in the stratified solids.

6. Waves in a three layered laminate—an example

Recently, Achenbach and Xu (1999) analyzed point-axisymmetric waves in a plate using elastodynamic reciprocity. An appropriate orthogonality relation with a dummy wave mode determines the coefficients of an expansion in Lamb wave modes of the wave motion generated in an infinite elastic layer by a point load normal to the faces of the plate. In this paper, a laminated plate of three layers with equal thickness h, shown in Fig. 4, is considered as an example. The top and bottom layers are made of the same material with Lame constants λ_1, μ_1 ($\lambda_1 = \mu_1$ in value), and mass density ρ_1. The middle layer is more rigid with $\lambda_2 = 2\lambda_1$, $\lambda_2 = \mu_2$, and $\rho_2 = \rho_1$. A vertical force of magnitude F_0 and Heaviside function of time, $-F_0 H(t)\delta(r)/2\pi r$, acts at a point of the top surface, $r_0 = 0$, $z_0 = 0$. Two receivers are set at points A (2.39 h, 0.05 h) and B

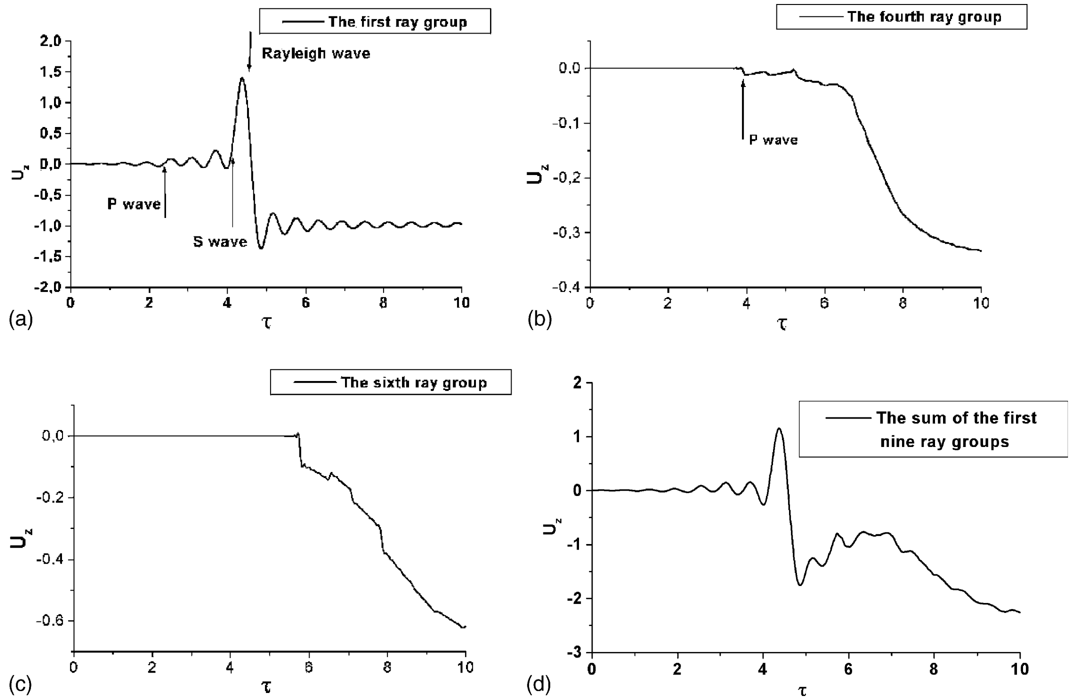

Fig. 5. Displacement u_z at receiver A: (a) the first ray group, (b) the fourth ray group, (c) the sixth ray group, (d) summation of first nine ray groups (more 7000 rays).

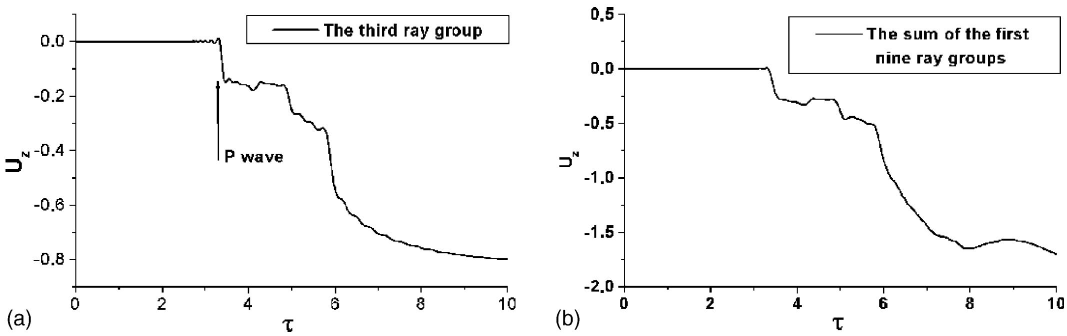

Fig. 6. Displacement u_z at receiver B: (a) the third ray group, (b) summation of first nine ray groups (more 7000 rays).

(2.39 h, 2.95 h) respectively. The responses that the normalized vertical displacement $U_z = u_z/(F_0/\mu_1 \pi h_1)$ varies with the normalized time $\tau = c_1 t/h$ are shown in Figs. 5 and 6, where $c_1^2 = (\lambda_1 + 2\mu_1)/\rho_1$.

Fig. 5a shows the first ray group directly from the source to the receiver A. The P- and S-wave arrive at the receiver A respectively when $\tau = 2.39$ and $\tau = 4.14$, the first peak appears at $\tau \approx 4.50$, which means the arrival of the Rayleigh surface wave. The result is similar to that shown by Pekeris (1955) who calculated the exact surface response of a half space due to a point force applied at the surface. Our result does not show the sharp rise and fall of the displacement at the arrival time of Rayleigh surface wave because the receiver A is slightly below the free surface. The response due to partial waves of the fourth group for R^3

and the sixth for R^5 are shown respectively in Fig. 5b and c. The paths of rays from the source to the receiver A are similar with those shown in Fig. 3c and e. Each ray is identified by the mode of each segment (P or S) and the layer (subscript 1, 2, or 3). The ray of the fourth group with two downward modes, p_1, p_2 and two upward modes P_2, P_1, arrives at the receiver A first at $\tau = 3.96$. Similarly, the sixth ray group contains 4×2^6 generalized rays with six wave segments from the source to the receiver A. The first arrival ray in this group is the ray with modes, $p_1p_2P_2p_2P_2P_1$, here P-wave speed in the layer 2 is twice of the P-wave speed in layers 1 and 3. The normalized arrival time is $\tau = 5.64$. The summation of the first nine groups is shown in Fig. 5d, which is the summation of more than 7000 generalized rays.

In the Fig. 6, the normalized displacement at the receiver B is shown. The Fig. 6a exhibits the third ray group arriving at B point, which includes eight rays from the source to the receiver B. The $p_1p_2p_3$ wave arrives at B point at $\tau = 3.43$. The summation of the first nine ray groups arriving at the point B contains almost 7000 rays.

For the displacement on or near the top surface of plate, the first ray group includes P-wave, S-wave and Rayleigh wave. The k-domain response for Rayleigh wave at a fixed frequency will asymptotes to a constant value when k approaches to infinity (Weaver et al., 1996), so the computational truncation by FFT and FHT will make some error for the first ray group of response on or near the top surface of plate. It is the reason that there exists a small signal before the arrival of the P wave in Fig. 5a. In the Fig. 6, the similar sinuousness in the response near the bottom does not appear.

7. Conclusion

In this paper we have extended the method of reverberation-ray matrix, which was developed originally for a framed structure (Howard and Pao, 1998) and modified for acoustic (dilatation, P wave) in layered liquid (Pao et al., 2000), to analyze elastic waves in layered solid. The additional consideration of distortional waves (S-waves) presents low difficulty in the mathematical formulation of the matrix. The final expressions for the transient waves in the entire medium are expressed in the form of double integrals over the frequency and transverse wave numbers (Eqs. (36)–(38)), and the integrand contains the matrix $[\mathbf{I} - \mathbf{R}]^{-1}\mathbf{s}$, where $\mathbf{R}$ is the reverberation matrix and $\mathbf{s}$ is a column matrix representing waves radiated from a point or a line source. Such a formulation enables one to expand the inverse of matrix $[\mathbf{I} - \mathbf{R}]$ into a power series of $\mathbf{R}$, and to convert the double integral with poles to a series of integrals, known as the ray-integrals. Each term of ray-integrals which is regular within the range of integration represents a group of partial waves transmitting from the source to a receiver along a multi-reflected ray-path (including P to S and S to P mode conversion). Each ray-integral in expansion can then be identified with the ray-integrals in series of generalized-rays, which could be evaluated precisely by applying the Cagniard's de-Hoop method (Pao and Gajewski, 1977). With the advance of digital computer and numerical method, the ray-integrals, a double integral in frequency and wave number, can be evaluated numerically by applying an algorithm (FHT) in inverse Hankel transform and another one (FFT) in inverse Fourier transform.

The major advantage of applying the reverberation-ray matrix is to have all partial waves that are propagating from the source to a receiver along all possible paths grouped systematically, and to have them sorted automatically into ordered groups according to the power of $\mathbf{R}^N$, the numbers of scatterings at interfaces of the layered solid. The systematic sorting of the rays groups in matrix form renders the method very effective in the computer programming for calculating large number of ray-integrals. This is illustrated by the example of a three-layered laminated plate under the impact of a vertical force at the surface. Transient responses at the opposite side of the plate are determined by numerically evaluating nine ray groups including $\mathbf{I}s, \mathbf{R}s, \mathbf{R}^2s, \ldots, \mathbf{R}^8s$. As large as 7000 number of rays are summed and then integrated numerically each time. It would be unthinkable to handle such a large number of generalized-ray integral by applying the Cagniard's de-Hoop method.

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